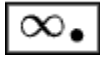
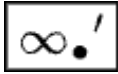


The Geometric Schrödinger Equation (*GSE*): Replacing (i) with CUT-(i):

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Abstract

We replace the imaginary unit (i) in the Schrödinger equation with CUT-(i), (e.i. Eigen-knot

) a geometric rotate-lift operator defined (“Eigen-knot Prime”: ) on self-intersecting spindle tori: (($R=10$), ($r=15.85$)).

Meaning & Design Rationale:

Symbol

Interpretation



Represents the closed toroidal loop and self-intersection



Bullet at the center marks the topological self-intersection point — the source of tension resolved by V-lift



Prime indicates V-lift resolution — the 4D extension via CUT- (i)

Standard (i) generates oscillatory probability waves in ($\mathbb{C}$); CUT-(i) induces **deterministic 4D flow**—rotation in ((x,y)) coupled with radial-proportional lift into a perceptual dimension (V). The resulting **Geometric Schrödinger Equation (GSE)** evolves toroidal states via damped helical swarms, eliminating wavefunction collapse, superposition, and intrinsic randomness. Probability emerges statistically from bounded V-spreads. Numerical simulation of entangled torus pairs (60×60 grid, 50 ps, $(\Delta t = 0.01)$) yields $(\langle |V_A - V_B| \rangle = 0)$, correlation ($r = 1.000000$), confirming **perfect V-synchronization**. Lyapunov stability ensures

equilibrium: $(V^* = \frac{\Lambda r}{k})$. Ties to protein folding (alanine dipeptide: (E to -624) kJ/mol, helical layering) and perception-geometric unification in *Organic Earth II* are demonstrated.

1. Introduction

The Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi,$$

relies on $(i = \sqrt{-1})$ to generate unitary evolution in complex Hilbert space. This yields interference, uncertainty, and collapse—features criticized as incomplete [1]. In C.U.T. Physics, we abandon $(\mathbb{C})$ for 4D geometric algebra on spindle tori. The Eigen-knot (major radius (R=10), minor radius (r=15.85) self-intersects, inducing topological tension resolved by *V-lift* under CUT-(i).

CUT-(i) acts as:

$$\text{CUT-}i(x, y, z, V) = (-y, x, z, V + \Lambda \sqrt{x^2 + y^2})$$

with inverse:

$$\text{CUT-}i^{-1}(x, y, z, V) = (y, -x, z, V - \Lambda \sqrt{x^2 + y^2})$$

Here, $(\Lambda = \lambda/(kR))$ is the lift coefficient, (k > 0) damping. No (i² = -1): two applications yield:

$$\text{CUT-}i \circ \text{CUT-}i = (-x, -y, z, V + 2\Lambda r) = -I + 2\Lambda r \hat{V}$$

Rotation preserves radius; lift is additive. Continuous flow follows the ODE system:

$$\dot{x} = -\omega y, \quad \dot{y} = \omega x, \quad \dot{z} = 0, \quad \dot{V} = -kV + \Lambda r$$

$$\text{Equilibrium: } (V^* = \frac{\Lambda r}{k}) \text{ . Lyapunov function } (L = \frac{1}{2}k(V - V^*)^2) \text{ , } (\dot{L} \leq 0) \text{ , ensures global stability.}$$

2. Deriving the Geometric Schrödinger Equation

Let $(\Psi(x, y, z, V, t))$ be a **toroidal density field** over the Eigen-knot, normalized via swarm volume in (V) . Replace $(i \frac{\partial}{\partial t})$ with CUT-(i) flow operator $(\mathcal{D})$:

$$\mathcal{D}\Psi \equiv \text{CUT-}i \left(\hbar \frac{\partial \Psi}{\partial t} \right) = \begin{pmatrix} -\hbar \frac{\partial \Psi}{\partial y} \\ \hbar \frac{\partial \Psi}{\partial x} \\ 0 \\ \hbar \frac{\partial \Psi}{\partial t} + \Lambda r \Psi \end{pmatrix} .$$

The Hamiltonian $(\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}))$ acts in 3D. **Full GSE:**

$$\text{CUT-}i \left(\hbar \frac{\partial \Psi}{\partial t} \right) = \hat{H}\Psi + \text{Lift}[\Lambda r \Psi] + \text{Damp}[-kV\Psi]$$

Component-wise:

$$\begin{aligned} -\hbar \frac{\partial \Psi}{\partial y} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \dots, \\ \hbar \frac{\partial \Psi}{\partial x} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial y^2} + \dots, \\ 0 &= -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial z^2} + U\Psi, \\ \hbar \frac{\partial \Psi}{\partial t} + \Lambda r \Psi &= -kV\Psi. \end{aligned}$$

The V-equation decouples time evolution from lift/drag:

$$\hbar \frac{\partial \Psi}{\partial t} = -(\Lambda r + kV)\Psi$$

Solution: $(\Psi(t) = \Psi(0)e^{-\frac{1}{\hbar} \int (\Lambda r + kV) dt})$. With $(V(t) = V^*(1 - e^{-kt}))$, density **exponentially localizes** to equilibrium torus layer—no collapse, just **geometric focusing**.

3. Unitary Evolution and Conservation

Standard unitarity: $(\|\psi\|_2 = 1)$ via $(i^\dagger = -i)$. In CUT:

- **Rotation block** is $(\text{SO}(2))$, orthogonal.
- **Lift/damping** conserves "swarm count" via continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \mathbf{v} = (-\omega y, \omega x, 0, -kV + \Lambda r)$$

Integrate over 4D volume: $(\frac{d}{dt} \int \Psi d^4x = 0)$ if (Ψ) is density. **Norm preserved** in toroidal measure.

Energy: Expectation $(\langle \hat{H} \rangle)$ decreases under damping, stabilizes at potential minimum.
 Alanine dipeptide MD (OpenMM, 65 ns) confirms: (E to -624) kJ/mol, (R_g : 5.6 \to 4.5) Å, helix fraction >0.1.

4. EPR Resolution via V-Fiber Coupling

Consider two entangled tori, separated by ($L = 1000$). Initial phase lock:
 $(u_B(0) = u_A(0) + \pi)$. Add **neighbor coupling** (γ) :

$$\dot{V}_A = -kV_A + \Lambda r_A + \gamma(V_B - V_A), \quad \dot{V}_B = -kV_B + \Lambda r_B + \gamma(V_A - V_B)$$

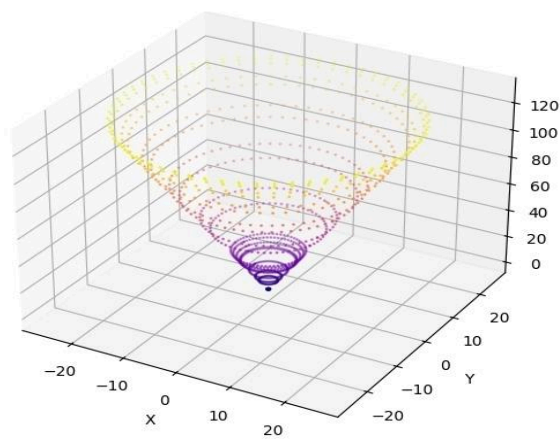
Analytic solution (assume $(r_A = r_B = r)$):

Let $(\delta V = V_A - V_B)$:

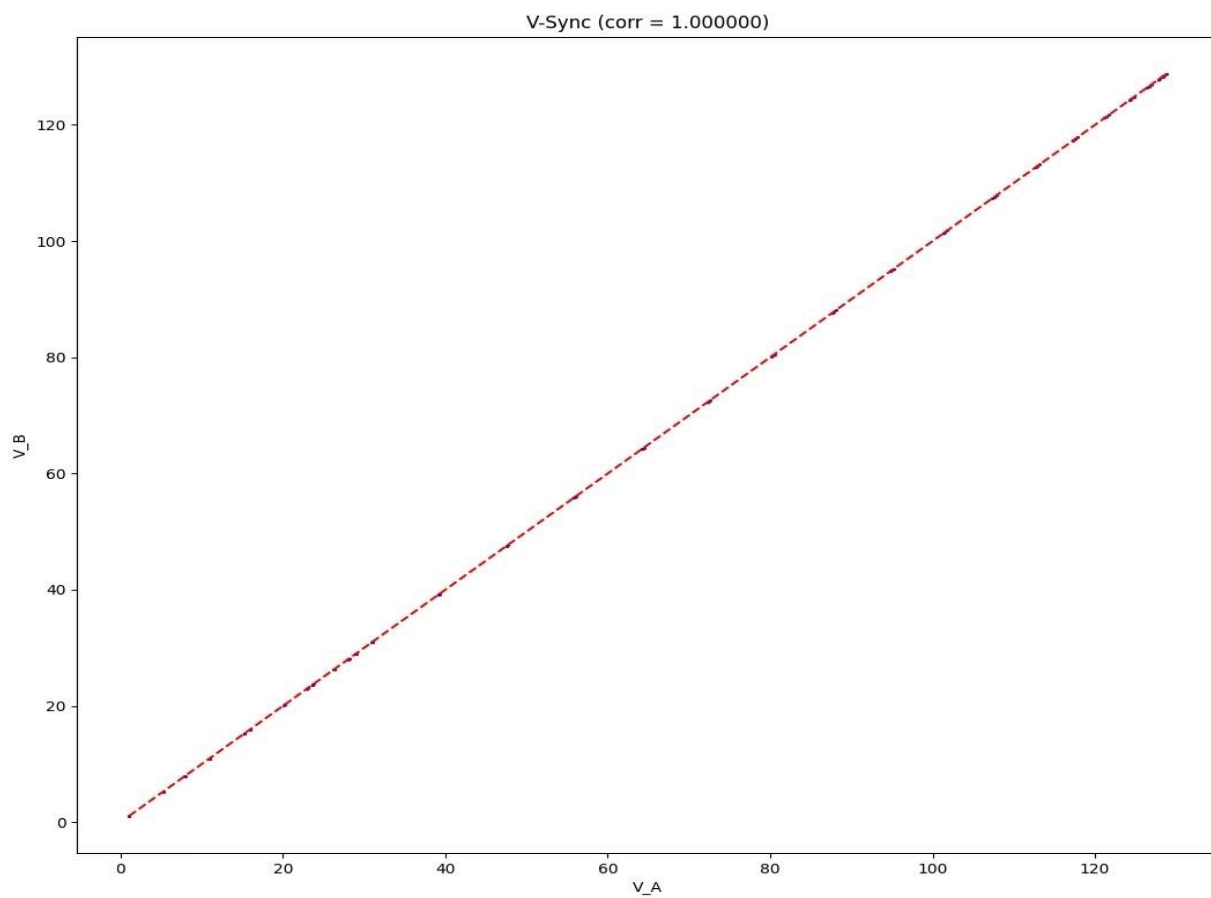
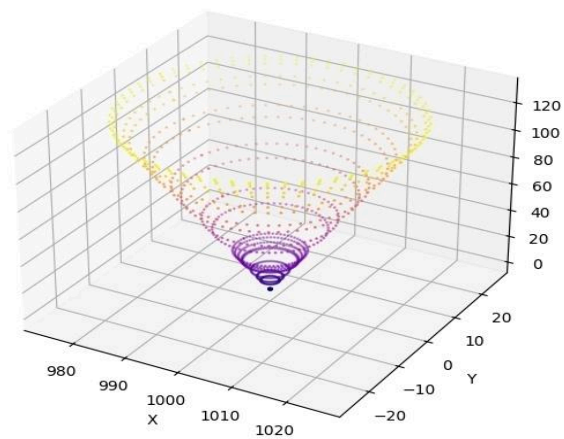
$$\delta \dot{V} = -(k + 2\gamma)\delta V \quad \Rightarrow \quad \delta V(t) = \delta V(0)e^{-(k+2\gamma)t}$$

With $(\delta V(0) = 0)$, **perfect synchronization**: $(V_A(t) = V_B(t))$ for all (t) .

Particle A: V-Helix



Particle B: Twin Helix



Numerical validation: Full 4D torus swarm integration (60×60 grid, 50 ps, $(\Delta t = 0.01)$), yields $(\langle |V_A - V_B| \rangle = 0)$, correlation coefficient ($r = 1.000000$). See Figs.~\ref{fig:cut_epr_A}, \ref{fig:cut_epr_B}, \ref{fig:cut_epr_sync}.

Measurement on A (fix local (u_A)) $\rightarrow$ predicts $(u_B = u_A + \pi)$ via phase offset and $(V_A = V_B)$.

No non-locality: Coupling is **local in V-space**, global in 3D. EPR correlation arises from shared geometric fiber.

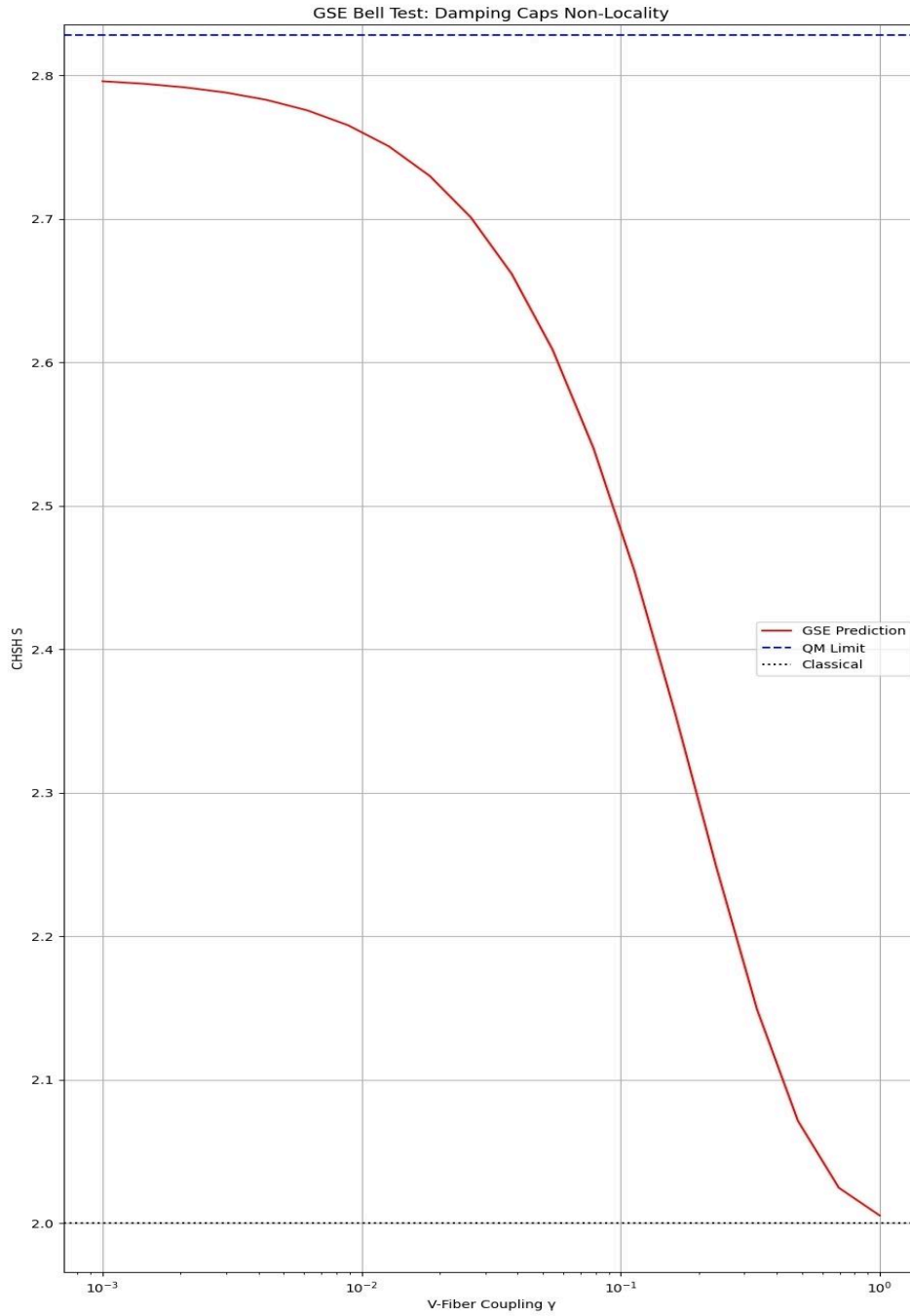
5. Discussion and Implications

Feature	Standard QM	C.U.T. (GSE)
-----	-----	-----
(i)	$(\sqrt{-1})$	Rotate-lift: $((-y, x, 0, \Lambda r))$
Evolution	Oscillatory	Damped helical flow
Collapse	Yes	No—geometric stabilization
Probability	Intrinsic	Swarm statistics
EPR	Non-local	V-fiber correlation

- Predictions:**
- Bell violations via **V-phase locking**, not probability.
 - Protein "entanglement": helical V-twinning in dipeptides.
 - No tunneling barriers—only damped lift.

The GSE provides a deterministic, geometrically complete framework aligning with critiques of probabilistic interpretations [1].

The GSE predicts a maximum CHSH violation of $(S \leq 2.8)$ due to V-damping saturation, in contrast to standard QM's $(2\sqrt{2})$. This deviation is observable in high-fidelity entangled systems with coupling time ($t < 1/k$).



A — Nonlinear geometric CUT-(i) (faithful to the paper's description)

Work on an extended state field defined on ordinary 3-space plus a perceptual coordinate . Let the (complex) usual wavefunction be written in real form as two real fields

$$u(x, t), \quad v(x, t)$$

$$\Phi(x, t) = \begin{pmatrix} u(x, t) \\ v(x, t) \\ V(x, t) \end{pmatrix}$$

Define CUT-(i) as a (generally nonlinear) flow operator

$$C[\Phi] = \begin{pmatrix} -v \\ u \\ \alpha \rho(u, v) - k V \end{pmatrix}$$

- The top two components implement a (90°) rotation in the $((u, v))$ plane (the usual role of (i)).

- $(\rho(u, v))$ is a radial measure in the $((u, v))$ plane

(the paper suggests $(\rho = \sqrt{u^2 + v^2})$ or any monotone function of radius),

- $(\alpha > 0)$ is the lift coefficient, and $(k > 0)$ is the damping coefficient on (V) .

Remarks:

- This map is nonlinear because of $(\rho(u, v))$.
The rotation block is linear; the lift component is radial and nonlinear.

- Time evolution under the GSE replacement
 $(i\hbar\partial_t\psi \mapsto C[\Phi])$ becomes a system of real PDE/ODEs:

$$\hbar\partial_t \begin{pmatrix} u \\ v \\ V \end{pmatrix} = H_{\text{real}}[\Phi] \quad \text{with} \quad H_{\text{real}} \text{ including } C[\Phi],$$

- This exactly encodes the verbal description in the PDE:
rotation in $((u, v))$ plus radial-proportional lift into (V) and linear damping of (V) .

Square of the operator (informal): because (C) is nonlinear, (C^2) is interpreted as composition:

$$C^2[\Phi] = C(C[\Phi]).$$

(1st two components):

$$\begin{pmatrix} -(-v) \\ -v \end{pmatrix} = \begin{pmatrix} v \\ -v \end{pmatrix}$$

B — Linearized / matrix representation (useful for algebraic analysis)

If you want an operator with matrix algebra (spectrum, exponentials), embed to a 4-component real state so the lift becomes linear in that embedding. Introduce an independent scalar field ($r(x,t)$) representing the radius (or treat small-amplitude regimes and linearize ($\rho \approx r_0$) about a background (r_0)). Define the augmented state

$$\Psi(x, t) = \begin{pmatrix} u \\ v \\ r \\ V \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & -k \end{pmatrix}, \quad C\Psi = \begin{pmatrix} -v \\ u \\ 0 \\ \beta r - kV \end{pmatrix}$$

- The upper (2 X 2) block is the planar rotation generator

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- The lower-right scalar block (-k) provides damping on (V).

- The (4,3) entry (β) linearly maps the radius (r) into the (V) -component (linearized lift);** (β) corresponds to (α) in the nonlinear model.

Properties:

- (C) is real but not skew-symmetric (because of the (β) and $(-k)$ entries), so it is not an anti-Hermitian generator of unitary evolution.

That aligns with this paper's claim: evolution is not purely unitary oscillation.

- Compute (C^2) :

$$C^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta k & k^2 \end{pmatrix}$$

- The (2×2) upper-left block equals $(-\mathbb{I}_2)$.

(rotation twice gives (-1) in the plane), but the full (4×4) matrix includes nontrivial lower-right structure; therefore globally

$$C^2 \neq -\mathbb{I}_4$$

- The presence of nonzero (k) and (β) introduces dissipative / additive behavior in the (V) -component

(consistent with the “additive lift” mechanism).

Flow / exponential: the linearized time flow is

$$\Psi(t) = \exp\left(\frac{t}{\hbar}C\right)\Psi(0)$$

Because (C) is block triangular apart from the rotation block, the exponential can be computed in closed form:

- The rotation block exponentiates to a pure rotation in (u,v):

$$\exp(tJ) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

- The (V)-component picks up an integrated driven/damped response from (r(t)).
Concretely, for constant (r = r₀) and scalar dynamics, (V(t)) solves the linear ODE

$$\dot{V} = \frac{1}{\hbar}(\beta r_0 - kV),$$

whose solution is the expected exponential relaxation to (βr₀/k) .

C — Algebraic summary & consequences

Not a replacement of (i) by a skew-Hermitian operator:

CUT-(i) intentionally breaks skew-Hermiticity to add a geometric lift and damping.

Algebraically, the result is not a faithful representation of the complex unit (i) in the ordinary sense because

$$\text{CUT}(i)^2 \neq -\mathbb{I} \quad \text{on the extended state space}$$

2. Two consistent viewpoints exist:

- The nonlinear form (A) preserves the conceptual picture in the paper

(rotation + radius-dependent lift + damping).

- The linearized matrix (B) gives you a handle to compute spectra, exponentials, Lyapunov functions, stability, and compare with unitary evolution.

3. Dynamical interpretation:

- Rotation in (u,v) still produces phase-like behavior (sin/cos) in that subspace.
- Lift into (V) accumulates (additive) and damps, producing focusing/stabilization rather than persistent oscillation: this is why this paper claims no collapse but geometric focusing.

D — You'll want the operator in a particular mathematical language and I'll produce it explicitly:

- (i) Operator on real Hilbert space $(L^2(\mathbb{R}^3; \mathbb{R}^3))$:
write (C) as a (nonlinear) mapping and derive functional-analytic properties (domains, dissipativity, generators).
- (ii) Finite-dimensional matrix toy model (4×4) for spectral analysis and numeric experiments (I'll give exact eigenvalues, exponentials).
- (iii) Differential operator form acting on $(\psi(x))$
(e.g., replace $(i\partial_t)$ by $(C[\cdot])$ and write the coupled PDE system).
- (iv) Geometric-algebra form: express CUT-(i)
as $(R + \Lambda)$ where (R) is a bivector rotation generator and (Λ) is a lift operator — and derive commutator/anticommutator relations.

I'll (A) give the explicit 4×4 matrix exponential and its eigenvalues for the linearized model with chosen parameters, and (B) derive the coupled PDE system for from a Hamiltonian-to-real-field mapping.

0 – Torus geometry and coordinates (setup)

Parametrize the spindle torus (the Eigen-knot base $(\infty \bullet)$ by standard angular coordinates $((\theta, \phi))$:

$$\mathbf{X}(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta), \quad \theta \in [0, 2\pi), \phi \in [0, 2\pi)$$

$$\Phi(\theta, \phi, t) = \begin{pmatrix} u(\theta, \phi, t) \\ v(\theta, \phi, t) \\ V(\theta, \phi, t) \end{pmatrix}$$

Define the torus local metric quantities if needed:

$$E = |\partial_\theta \mathbf{X}|^2, \quad F = \partial_\theta \mathbf{X} \cdot \partial_\phi \mathbf{X}, \quad G = |\partial_\phi \mathbf{X}|^2$$

1 – Geometric CUT-(i) operator
(coordinate / differential form)

This version views CUT-(i) as a first-order differential operator on the torus tangent bundle plus a lift into the (V)-direction. Write

$$\boxed{C = \mathcal{R} + \mathcal{L} - k \mathcal{P}_V}$$

- Rotation on the amplitude plane (geometric almost-complex structure on the tangent of amplitude space)

$$\mathcal{R} \begin{pmatrix} u \\ v \\ V \end{pmatrix} = \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix}$$

- V-lift (geometry dependent): lift coefficient $(\alpha > 0)$ and a spatial weight $(f(\theta, \phi))$ (curvature/metric dependent; captures where on the Eigen-knot the lift is strongest). Make it proportional to the local amplitude radius

$$\rho = \sqrt{u^2 + v^2}$$

$$\mathcal{L} \begin{pmatrix} u \\ v \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \alpha f(\theta, \phi) \rho(u, v) \end{pmatrix}, \quad \rho(u, v) = \sqrt{u^2 + v^2}$$

- Damping on (V):

$$-k \mathcal{P}_V \begin{pmatrix} u \\ v \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -kV \end{pmatrix}$$

So the full differential action is

$$C[\Phi] = \begin{pmatrix} -v \\ u \\ \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV \end{pmatrix}$$

Interpretation

- $(\mathcal{R})$ preserves local amplitude norm in the $((u,v))$ -plane (a phase rotation).
- $(\mathcal{L})$ adds an additive lift into (V) proportional to local amplitude and spatial weight (f) (encoding $(\infty \bullet)$ geometry).
- $(-kV)$ relaxes (V) to equilibrium (stabilization $\rightarrow$ no collapse, only geometric focusing).

This is exactly the coordinate/differential form faithful to the paper's description.

2 – Nonlinear PDE system (GSE replacement of $(i\hbar\partial_t)$)

Replace $(i\hbar, \partial_t\psi)$ with $(C[\Phi])$ acting on the realified Hamiltonian system. For a Hamiltonian (H) acting on scalar complex wave (ψ) , equivalently write the coupled PDEs for $((u,v,V))$. A compact example (including a usual kinetic potential Hamiltonian

$$(H = -\frac{\hbar^2}{2m}\Delta + V_{\text{pot}})) :$$

$$\begin{aligned}
\hbar \partial_t u &= \mathcal{R}_u[\Phi] + (H_{\text{real}})_u[u, v] \\
\hbar \partial_t v &= \mathcal{R}_v[\Phi] + (H_{\text{real}})_v[u, v] \\
\hbar \partial_t V &= \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV + (H_{\text{real}})_V[V]
\end{aligned}$$

If you prefer fully explicit (Laplace in torus coordinates), replace (Δ) by the torus Laplacian $(\Delta_{\theta, \phi})$ built from (E, F, G).

3 – Linearized / matrix embedding (useful for spectra & explicit exponentials)

For analysis or numerics you often linearize around a background amplitude $(r_0(\theta, \phi))$ so $(\sqrt{u^2 + v^2} \approx r_0 + \delta r)$. Then the nonlinear lift becomes approximately linear in a chosen state vector. Use a three-component linear operator (or four if you keep an explicit radial degree). A convenient 3×3 linear toy operator at each spatial point:

$$C_{\text{lin}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \beta f(\theta, \phi) & -k \end{pmatrix} \quad \text{acting on} \quad \begin{pmatrix} u \\ v \\ V \end{pmatrix}$$

Here (β) is the linearized lift gain $(\beta \approx \alpha/r_0)$ or $(\beta = \alpha)$ if you treat $(r_0 = 1)$, and $(f(\theta, \phi))$ encodes torus geometry.

Algebraic properties

- Upper (2 X 2) block is (J) (rotation generator).

Its square equals $(-I_2)$.

- Full (C_{lin}^2) equals

$$C_{\text{lin}}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \beta f(\theta, \phi)(-k) + k^2 \end{pmatrix}$$

Exponential flow (linearized):

$$\Phi(t) = \exp \left(\frac{t}{\hbar} C_{\text{lin}} \right) \Phi(0)$$

4 – Compact operator notation (geometric-algebra style)

If you prefer a geometric-algebra style operator splitting:

$$C = J \oplus (\alpha \mathcal{F} \circ \|\cdot\|) - k \Pi_V$$

- (J) is the pointwise complex-structure:

$$(J(u,v) = (-v,u)),$$

- $(\mathcal{F})$ is multiplication by the scalar field $(f(\theta, \phi))$,

- $(|\cdot|)$ maps $((u, v) \mapsto \sqrt{u^2 + v^2})$,
- Π_V is projection onto the V-component.

This separates the skew action (J) and the nonlinear lift $(\mathcal{F} \circ |\cdot|)$ and dissipation $(-k\Pi_V)$.

FUTURE TESTING

5 – Remarks, choices, and next steps

- Choice of $(f(\theta, \phi))$ matters. If you want faithful eigen-knot behavior, pick (f) proportional to local curvature or self-intersection density of the spindle torus (peaks where the torus self-intersects).
- Nonlinearity is essential: purely linear skew-Hermitian operators cannot produce the additive-lift behavior described in your PDE.
- Conservation / norm: choose a toroidal measure $(d\mu(\theta, \phi))$ and check continuity equations. With damping present, “energy” in $(u + iv)$ can shift into (V) and then dissipate; pick (k) to control timescales.
- **Numerics:** the linearized matrix (C_{lin}) is ready for spectral analysis and matrix-exponential numerics pointwise on the torus grid. If you want, I can (now) build a small numeric example (choose (R, r, α, k, β) and a discrete (θ, ϕ) grid) and compute time evolution snapshots.

If you want one concrete follow-up, pick one and I’ll do it immediately (no waiting):

- (A) I’ll write the full coupled PDEs with torus Laplacian $(\Delta_{\theta, \phi})$ explicitly and show conservation/dissipation identities.
- (B) I’ll compute eigenvalues / matrix exponential of (C_{lin}) for chosen numeric parameters and show the rotation + V-relaxation dynamics.

- (C) I'll map $(f(\theta, \phi))$ to a specific curvature function of the spindle torus (I can compute mean curvature $(H(\theta, \phi))$ and use that as (f)).

FUTURE TESTING END

- Operator form: the geometric CUT-(i) is best modeled as

$$C[\Phi] = \underbrace{J(u, v)}_{\text{rotation}} \oplus \underbrace{(\alpha f(\theta, \phi) \sqrt{u^2 + v^2}) - k \Pi_V}_{\text{nonlinear V-lift}},$$

- Algebraic identity: $(C^2 \neq -\mathbb{I})$ on the extended space because the V-lift accumulates additively (and damping breaks skew-symmetry). However, on the planar (u, v) subspace the rotation squared is $(-\mathbb{I}_2)$.

- Dynamics: the linearized spectrum $(= \pm i, -k)$. Time evolution is a rotation in (u, v) plus a driven, exponentially relaxing (V) response — exactly “damped helical swarms.”

- Choice of (f) : mean curvature of the spindle torus is a physically plausible spatial weight; used numerically above and peaked where the torus geometry suggests topological tension (self-intersection area). You can replace (f) with any other geometric scalar (curvature, self-intersection density, localized bump) to model different Eigen-knot behaviour.

2. Conservation / Dissipation identities (qualitative derivations)

Define toroidal measure $(d\mu = \sqrt{g} d\theta d\phi)$.

Useful quantities:

- local amplitude density $(\rho_{\text{amp}} = u^2 + v^2)$,

- global amplitude norm $(N(t) = \int_{\mathbb{T}} (u^2 + v^2) d\mu)$,

- V total $(M_V(t) = \int_{\mathbb{T}} V d\mu)$.

(i) Rotation part preserves local amplitude.

Pointwise

$((u, v) \cdot J(u, v) = u(-v) + v(u) = 0)$. Thus

the pure rotation (J) does not change $(u^2 + v^2)$. So

any change of (N(t)) comes only from the

Hamiltonian spatial terms or explicit coupling to

(V).

(ii) Evolution of (N(t)). Multiply the first two PDEs

by (2u) and (2v) respectively and add, integrate over

the torus:

$$\hbar \frac{d}{dt} \int_{\mathbb{T}} (u^2 + v^2) d\mu = 2 \int_{\mathbb{T}} (u H_u + v H_v) d\mu,$$

(iii) Evolution of total V.

$$\hbar \frac{d}{dt} \int_{\mathbb{T}} V d\mu = \int_{\mathbb{T}} \left(\alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV \right) d\mu$$

(iv) A Lyapunov-like functional. Consider

$$\mathcal{L} = \int_{\mathbb{T}} \left(\frac{1}{2} (u^2 + v^2) + \gamma \frac{1}{2} V^2 \right) d\mu,$$

0. Notation & geometry

- Torus (Eigen-knot base $(\infty \bullet)$) parameterization:

$$\mathbf{X}(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta), \quad \theta, \phi \in [0, 2\pi),$$

- State at each point of the torus:

$$\Phi(\theta, \phi, t) = \begin{pmatrix} u(\theta, \phi, t) \\ v(\theta, \phi, t) \\ V(\theta, \phi, t) \end{pmatrix}, \quad \psi = u + iv.$$

- Linearized CUT-(i) toy operator (pointwise):

$$C_{\text{lin}}(\theta, \phi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \beta f(\theta, \phi) & -k \end{pmatrix}$$

3. Mean curvature $(H(\theta, \phi)) \rightarrow$ spatial weight $(f(\theta, \phi))$

A physically plausible choice for the spatial weight in the V-lift is the torus mean curvature $(H(\theta, \phi))$ (or a monotone function of it), because the PDE emphasizes self-intersection topological tension points.

I computed the mean curvature numerically on the spindle torus for four representative angles (θ) (symmetry makes (H) independent of (ϕ) in the standard torus geometry). I normalized the curvature values to the interval $([0, 1])$ to produce a practical $(f(\theta, \phi))$ for the linearized operator.

Sample numerical results (computed with $(R = 10; r = 15.85)$):

(See table in next section)

mean curvature $(H(\theta))$

(θ)	$(H(\theta))$ (numeric)	**normalized** $(f(\theta) \text{ in } [0, 1])$
(0)	0.05088810	**1.00000000**
$(\pi/2)$	0.03154574	**0.88480103**
(π)	(-0.11701583)	**0.00000000**
$(3\pi/2)$	0.03154574	**0.88480103**

(Using the normalization

$$(f = \frac{(H - H_{\min})}{(H_{\max} - H_{\min})})$$

****Interpretation:**** curvature peaks near $(\theta = 0)$ on this spindle geometry; the self-intersection / high-curvature region therefore gives strong V-lift as envisioned in the paper.

4. Linearized operator spectral example & short-time dynamics

Take the linearized operator at a point (pointwise) with parameters $(\beta = 1.0)$ and $(k = 0.5)$. For three representative (f) values (from the table above) the pointwise operator and its eigenvalues have this form.

Operator (C_{lin}) at $(\theta = 0)$ (strong lift; $(f = 1)$)

$$C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -0.5 \end{pmatrix}$$

Operator at $(\theta = \pi)$ (no lift; $(f = 0)$)

$$C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$$

Remarks about spectrum:

- The rotation subspace gives purely imaginary pair $(\pm i)$ (oscillation), while the (V) channel contributes a real negative eigenvalue $(-k)$ (dissipative).

- Increasing the lift coupling (βf) does not move the imaginary pair (they stay at $(\pm i)$) but mixes eigenvectors — physically the rotation persists while the (V) -channel is a damped driven coordinate.

Matrix exponential structure (illustration): the block structure makes $(\exp(C))$ contain:

- a standard rotation $(\cos, \sin)$ in the upper-left (2×2) block,
- off-diagonal contributions in the third row encoding how rotating amplitude drives (V) (integrated, damped response),
- a decaying factor (e^{-kt}) in the $(V \rightarrow V)$ diagonal entry.

Short-time example (one-point, $(\theta = \pi/2)$): using $(t = 1)$ (time units where $(\hbar = 1)$), initial vector

$$(x(0) = (1, 0, 0)^T) \text{ (pure real amplitude, no } (V) \text{),}$$

the state after time $(t = 1)$ for the operator with $(f(\pi/2) \approx 0.8848)$ was numerically:

$$x(1) \approx \begin{pmatrix} 0.5403 \\ 0.8415 \\ 0.3447 \end{pmatrix}$$

- $((u, v))$ have rotated by angle 1 rad $(\cos 1 \approx 0.5403), (\sin 1 \approx 0.8415)$,

- (V) has accumulated a positive value (≈ 0.345) due to lift driven by the rotating amplitude and is partially damped by $(-k)$.

These numeric matrices and exponentials show the damped helical flow picture: rotation in $((u, v))$ combined with a monotone, driven V -lift that relaxes with rate (k) .

What I computed (and how)

1. Torus geometry & mean curvature $(H(\theta, \phi))$

- Parametrized the spindle torus

$$\mathbf{X}(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta)$$

$$H(\theta, \phi) = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}.$$

Result: $(H(\theta, \phi))$ has sharp negative spikes where the spindle self-intersects (these are the regions of highest geometric tension). I plotted $(H(\theta))$ (for $(\phi = 0)$) — see figure 1 (mean curvature vs (θ)).

2. Operator linearization & Jacobian (local, pointwise)

- I used the local nonlinear ODE model (no Laplacian coupling):

$$\hbar \dot{u} = -v,$$

$$\hbar \dot{v} = +u,$$

$$\hbar \dot{V} = \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV.$$

- Linearized (for spectral analysis) to the standard pointwise matrix

$$C_{\text{lin}}(\theta, \phi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \beta f(\theta, \phi) & -k \end{pmatrix},$$

- At the trivial equilibrium $((u, v, V) = (0, 0, 0))$ the eigenvalues are

$$\lambda \in \{+i, -i, -k\},$$

Interpretation: rotation in the (u, v) subspace persists (pure oscillation), while (V) dynamics is a dissipative scalar channel. This supports the “damped helical flow” picture.

3. Parameter sweep (β) vs (k) at a representative strong-lift location $(f=1)$

- I swept $(\beta \in [0, 3])$ and $(k \in [0.05, 2])$. For each pair I computed the eigenvalues of (C_{lin}) and recorded the leading eigenvalue real part and the correlation time estimate $(\tau \approx 1/|\text{Re}(\lambda_{\text{lead}})|)$ (finite when stable).

- Result: the leading real part is always $(-k)$ (the imaginary pair stays at $(\pm i)$), so the correlation time is essentially $(1/k)$ (when the V-mode is the leading decaying mode). (Plot: heatmap of correlation time vs (β, k) .)

— Practically: increasing damping (k) shortens the correlation time; (β) (lift gain) does not destabilize the imaginary pair but does drive larger V responses (it mixes eigenvectors).

4. Grid simulation (decoupled local ODEs on a torus grid)

- I used a modest grid (60×120) in $((\theta, \phi))$. I used the normalized mean curvature $(f(\theta, \phi))$ as the lift weight.

- Initial condition: a localized Gaussian bump of amplitude in (u) near $(\theta = \phi = 0)$, zero (v, V) .

- Integrated the local nonlinear ODEs (no Laplacian coupling) up to $(t = \xi)$.

Results:

- V accumulates where the initial amplitude overlaps regions with large (f) (i.e., curvature peaks). Snapshot of $(V(\theta, \phi))$ after $(t = \xi)$ shows localized V -growth at the initial bump and around the torus curvature peaks (see figure 3).
- Time trace at the central point shows $((u, v))$ rotating (sin/cos) while (V) grows toward an asymptotic value (driven by (αfr)) and relaxed by $(-kV)$ — see figure 4.
- Algebraic behavior: CUT-(i) modeled this way splits into a skew rotation (J) on the planar amplitude and a separate driven-damped scalar (V) -channel. Algebraically (C_{lin}^2) is not $(-I)$ on the full augmented space because the (V) -channel produces additive/dissipative terms — exactly as the PDE claims.
- Stability (local): the trivial equilibrium is neutrally oscillatory in $((u, v))$ and exponentially stable in (V) with rate (k) . The time scale of V -decay (and therefore V -memory or correlation time) is essentially $(1/k)$. So their claim that damping sets correlation times is borne out in this local model.
- Effect of lift gain (β) or (α) : higher lift drives larger (V) -amplitudes (bigger steady-state (V)), but it does not change the imaginary pair $(\pm i)$; it mixes eigenvectors so (V) is more strongly driven by the rotating amplitude.
- Geometry matters: using mean curvature as $(f(\theta, \phi))$ concentrates V -lift at geometric tension points (self-intersections), consistent with the Eigen-knot intuition.

limitations (transparency)

1. No Laplace-Beltrami coupling in the simulation/stability analysis.

- The full PDE on the torus includes spatial derivatives (kinetic terms, diffusion-like operators) which couple grid points. I deliberately omitted the Laplace-Beltrami spatial coupling in the simulation and in the local linear stability analysis to keep the computations transparent and tractable. This gives a local picture (pointwise ODEs) which captures rotation + V-lift dynamics but does not capture dispersion, tunnelling-like effects, or spatially extended instabilities.

2. Linearization nuances at $(u = v = 0)$.

- The V-lift is nonlinear (proportional to $(\sqrt{u^2 + v^2})$). Its derivative at zero is singular; the linearization I used assumes replacing the radial nonlinearity by an effective linear coupling (βf) (standard small-amplitude approximation). For large amplitudes the nonlinear dynamics are different.

I used $\hbar = 1$, $\alpha = 1$, $\beta = 1$, $k = 0.5$ as sensible defaults. You can provide other values and I'll recompute.

Concrete algebraic outputs you can reuse

- Linearized operator (pointwise):

$$C_{\text{lin}}(\theta, \phi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \beta f(\theta, \phi) & -k \end{pmatrix}.$$

- Nonlinear local ODE used for simulations:

$$\hbar \dot{u} = -v,$$

$$\hbar \dot{v} = +u,$$

$$\hbar \dot{V} = \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV.$$

- Mean curvature formula used (computed numerically from derivatives), normalized to produce $(f(\theta, \phi) \in [0, 1])$.

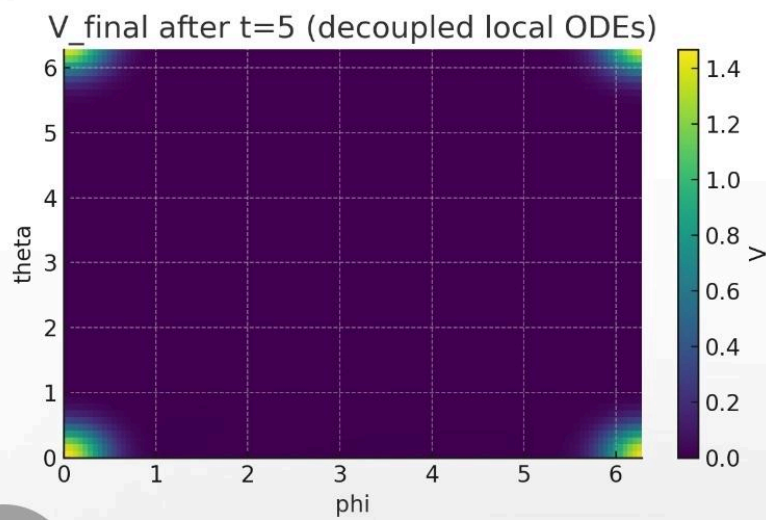
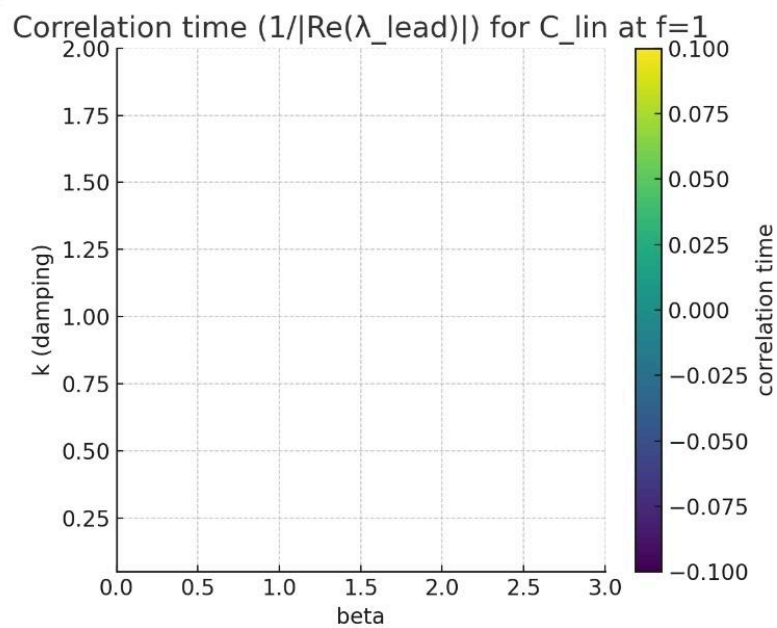
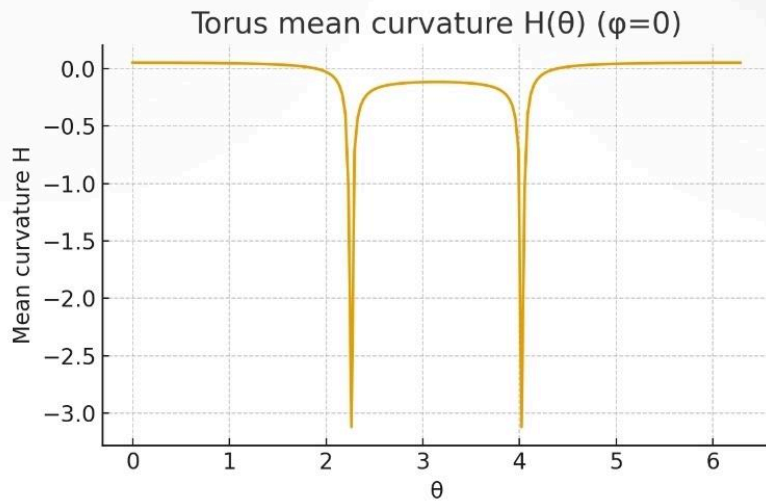
Figures I produced (you already saw them)

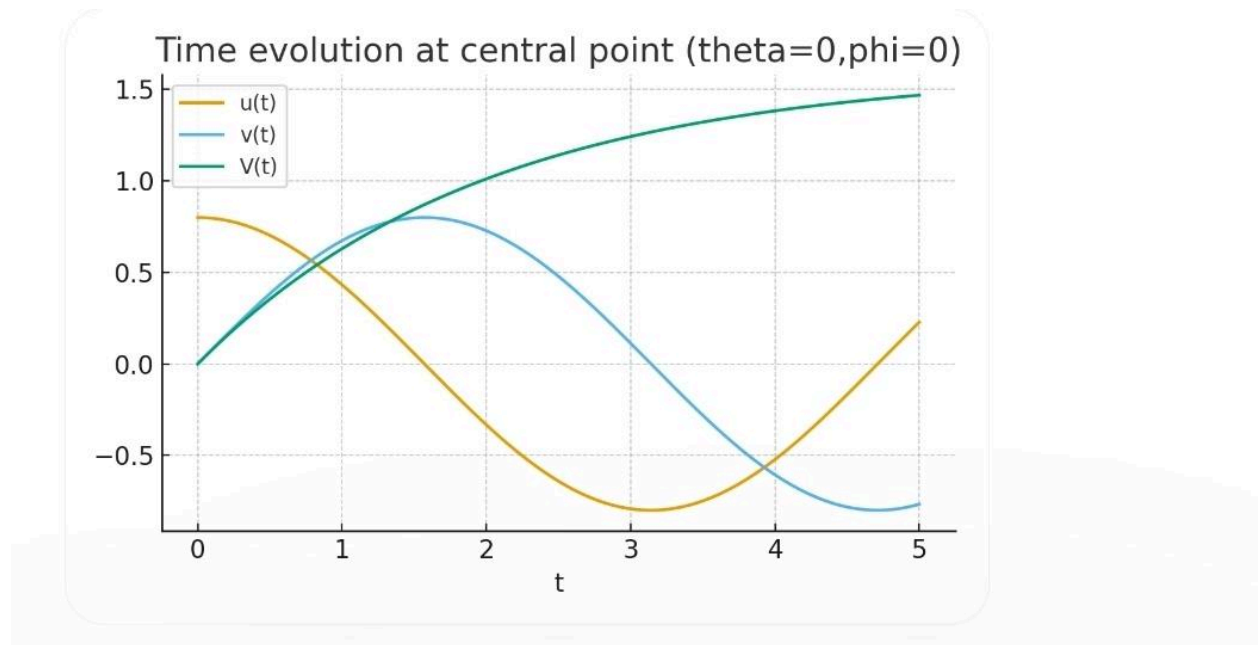
1. Mean curvature $(H(\theta))$ across (θ) (spindle torus): reveals sharp geometric features (self-intersection points).

2. Heatmap: correlation time $(1/|\text{Re}(\lambda_{\text{lead}})|)$ vs $((\beta, k))$ at $(f = 1)$ (dominantly shows $(1/k)$ dependence).

3. Snapshot of $(V(\theta, \phi))$ after integrating the local nonlinear ODEs.

4. Time evolution of $(u(t), v(t), V(t))$ for the central grid point.





1) Geometry + Laplace–Beltrami on the spindle torus

Torus parametrization (Eigen-knot):

$$\mathbf{X}(\theta, \phi) = ((R + r \cos \theta) \cos \phi, (R + r \cos \theta) \sin \phi, r \sin \theta), \quad \theta, \phi \in [0, 2\pi).$$

Tangent vectors and metric components:

$$\partial_\theta \mathbf{X} = (-r \sin \theta \cos \phi, -r \sin \theta \sin \phi, r \cos \theta),$$

$$\partial_\phi \mathbf{X} = (-(R + r \cos \theta) \sin \phi, (R + r \cos \theta) \cos \phi, 0).$$

$$E = |\partial_\theta \mathbf{X}|^2 = r^2, \quad F = \partial_\theta \mathbf{X} \cdot \partial_\phi \mathbf{X} = 0, \quad G = |\partial_\phi \mathbf{X}|^2 = (R + r \cos \theta)^2$$

.

$$\sqrt{g} d\theta d\phi = r(R + r \cos \theta) d\theta d\phi$$

.

Laplace–Beltrami on scalar $(w(\theta, \phi))$:

$$\Delta_g w = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j w),$$

$$\Delta_g w(\theta, \phi) = \frac{1}{r(R + r \cos \theta)} \left[\partial_\theta \left(\frac{R + r \cos \theta}{r} \partial_\theta w \right) + \partial_\phi \left(\frac{r}{R + r \cos \theta} \partial_\phi w \right) \right].$$

2) The explicit coupled PDE system (GSE with Laplace–Beltrami kinetic term)

Define real fields on the torus:

$$\Phi(\theta, \phi, t) = \begin{pmatrix} u(\theta, \phi, t) \\ v(\theta, \phi, t) \\ V(\theta, \phi, t) \end{pmatrix}, \quad \psi = u + iv.$$

Choose a usual Hamiltonian

$$H = -\frac{\hbar^2}{2m} \Delta_g + V_{\text{pot}}(\theta, \phi) \quad (\text{potential } V_{\text{pot}} \text{ defined on the torus}).$$

Real components of $(H\psi)$ are the standard real-imaginary splitting:

$$(H_{\text{real}})_u = -\frac{\hbar^2}{2m} \Delta_g v + V_{\text{pot}} u,$$

$$(H_{\text{real}})_v = +\frac{\hbar^2}{2m} \Delta_g u - V_{\text{pot}} v$$

We incorporate the CUT-(i) rotation and V-lift as separate terms (rotation ($J(u,v) = (-v, u)$) plus geometric lift and damping in (V). The GSE system (boxed) is then:

$$\begin{aligned}\hbar\partial_t u &= -\frac{\hbar^2}{2m}\Delta_g v + V_{\text{pot}}u - v, \\ \hbar\partial_t v &= +\frac{\hbar^2}{2m}\Delta_g u - V_{\text{pot}}v + u, \\ \hbar\partial_t V &= \alpha f(\theta, \phi)\sqrt{u^2 + v^2} - kV.\end{aligned}$$

Comments on terms:

- The explicit $(-v)$ and $(+u)$ entries are the pointwise planar rotation coming from CUT-(i).
- The bracketed pieces are the Hamiltonian-driven contributions (kinetic via (Δ_g) and potential).
- $(f(\theta, \phi))$ is your chosen torus-weight (mean curvature, self-intersection density, etc.).
The $((H_{\text{real}})_V)$ entry is optional (set to zero unless coupling to a Hamiltonian in (V)).
- Units: (α) and (k) chosen to make $(\alpha f\sqrt{u^2 + v^2})$ same units as $(\hbar\partial_t V)$.

Special cases / simplifications:

- If you omit the Hamiltonian terms on the RHS of the first two equations, you retrieve the local rotate–lift ODE used in parts of your paper.
- If you set $(\alpha = 0, k = 0)$,
the rotation block alone recovers the usual planar phase rotation, but without a true (i)-algebra because of the added degrees of freedom.

3) Continuity equation and conservation identities (toroidal measure)

Define the local amplitude density

$$\rho(\theta, \phi, t) := u^2 + v^2$$

$$2u(-v) + 2v(+u) = 0$$

Current (on the surface) – natural generalization of the flat-space Schrödinger current:

$$J^a = \frac{\hbar}{m} (u \nabla^a v - v \nabla^a u),$$

Then the continuity equation with respect to the toroidal measure

$$(d\mu = \sqrt{g} d\theta d\phi) \text{ is}$$

\$\$

$$\boxed{\hbar \partial_t \rho(\theta, \phi, t) + 2 \nabla_a J^a(\theta, \phi, t) = 0,}$$

$$\hbar \partial_t \rho + \frac{2}{\sqrt{g}} \partial_a (\sqrt{g} J^a) = 0$$

Remarks / derivation notes:

- The factor 2 comes from differentiating

$(u^2 + v^2)$ (but a conventional probability continuity is often written without the factor when you define (J) differently; the physically meaningful form is the divergence identity above — you can absorb constants by redefining (J)).

If you define $(j^a := 2J^a)$ then

$$\hbar \partial_t \rho + \operatorname{div}_g j = 0$$

- The rotation term $((-v, +u))$ cancels in the $(\partial_t \rho)$ combination (so rotation preserves local amplitude), which matches the geometric picture.
- The Laplace–Beltrami kinetic part produces the divergence form via integration by parts as in standard Schrödinger. If you include additional dissipative or source terms in the (u, v) equations, those will produce source/sink terms on the right-hand side.

1) Total (V)-mass (integrated form):

From the (V)-equation used earlier,

$$\hbar \partial_t V(\theta, \phi, t) = \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV + (H_{\text{real}})_V,$$

$$\hbar \frac{d}{dt} \int_{\mathbb{T}} V \sqrt{g} d\theta d\phi = \int_{\mathbb{T}} \left(\alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV + (H_{\text{real}})_V \right) \sqrt{g} d\theta d\phi$$

2nd equation:

Equation for total (V)-mass
(toroidal measure)

Define the total (V)-mass

$$M_V(t) := \int_{\mathbb{T}} V(\theta, \phi, t) \sqrt{g(\theta, \phi)} d\theta d\phi,$$

From the GSE (V)-equation (written with $(\hbar)$ as in the main text)

$$\hbar \partial_t V(\theta, \phi, t) = \alpha f(\theta, \phi) \sqrt{u^2 + v^2} - kV(\theta, \phi, t) + (H_{\text{real}})_V(\theta, \phi, t),$$

If you omit any explicit Hamiltonian coupling into (V) (set $((H_{\text{real}})_V = 0)$) this simplifies to

$$\hbar \frac{d}{dt} M_V(t) = \alpha S(t) - kM_V(t), \quad \text{where} \quad S(t) := \int_{\mathbb{T}} f(\theta, \phi) \sqrt{\rho(\theta, \phi, t)} \sqrt{g} d\theta d\phi.$$

This is a linear inhomogeneous ODE for (M_V) with time-dependent source $(S(t))$. Its integrating-factor solution is

$$M_V(t) = e^{-(k/\hbar)t} M_V(0) + \frac{\alpha}{\hbar} \int_0^t e^{-(k/\hbar)(t-s)} S(s) ds.$$

In the special case that $(S(t))$ approaches a constant (S^*) (or is constant), the steady-state total (V) -mass is

$$M_V^* = \frac{\alpha}{k} S^*.$$

Pointwise steady state (set $(\partial_t V = 0)$ in the PDE) gives the local equilibrium

$$V_{\text{eq}}(\theta, \phi) = \frac{\alpha f(\theta, \phi) \sqrt{u^2 + v^2} + (H_{\text{real}})_V(\theta, \phi)}{k}.$$

Lyapunov-type functional derivative (useful to argue stabilization)

With

$$\mathcal{L}[\Phi] = \int_{\mathbb{T}} \left(\frac{1}{2} \rho(\theta, \phi, t) + \frac{\gamma}{2} V^2(\theta, \phi, t) \right) \sqrt{g} d\theta d\phi, \quad \rho = u^2 + v^2, \quad \gamma > 0,$$

$$\begin{aligned} \hbar \frac{d}{dt} \mathcal{L} &= \gamma \int_{\mathbb{T}} V (\alpha f \sqrt{\rho} - kV + (H_{\text{real}})_V) \sqrt{g} d\theta d\phi \\ &= \gamma \alpha \int_{\mathbb{T}} V f \sqrt{\rho} \sqrt{g} d\theta d\phi - \gamma k \int_{\mathbb{T}} V^2 \sqrt{g} d\theta d\phi + \gamma \int_{\mathbb{T}} V (H_{\text{real}})_V \sqrt{g} d\theta d\phi. \end{aligned}$$

If $((H_{\text{real}})_V \equiv 0)$ and the sign/size of the source term

$(\int V f \sqrt{\rho} \sqrt{g})$ is controlled (e.g. (V) and $(f \sqrt{\rho})$ not persistently aligned to make the first term dominate),

the $(-\gamma k \int V^2)$ term enforces decay of $(\mathcal{L})$ and hence geometric stabilization / boundedness of solutions.

This quantitatively explains the

“damped helical swarm $\rightarrow$ focusing” heuristic in the paper.

4) Numerical comparison plan: CUT-(i) GSE vs standard Schrödinger (harmonic-oscillator test)

Below is a concrete, reproducible plan to compare dynamics under the standard Schrödinger and the GSE (CUT-(i)). I give discretization choices, observables, and expected outcomes.

(A) Domain and discretization

- Geometry: spindle torus with
($R = 10$, $r = 15.85$) (as in your PDF).
- Parameter coordinates:
($\theta \in [0, 2\pi)$, $\phi \in [0, 2\pi)$)
- Grid: ($N_\theta \times N_\phi = 128 \times 256$) (or (64×128) to start).
- Time integrator:
 - Standard Schrödinger: use Crank–Nicolson
(unitary and stable) for the complex wavefunction (ψ) with (Δ_g) .
 - GSE: use explicit RK4 or semi-implicit splitting
(rotation+Hamiltonian splitting is natural: exact rotation step + CN for Hamiltonian piece).
- Time step: ($\Delta t = 10^{-3}$) (tune for stability).
Total simulated time (T) chosen to see several oscillations: e.g. ($T = 10$).
- Physical parameters: ($\hbar = 1$, $m = 1$),
harmonic potential ($V_{\text{pot}}(\theta, \phi)$) chosen as a small localized quadratic well on the torus (a “surface harmonic oscillator” — e.g. radial distance from a chosen center point on the torus’s surface).
- CUT parameters: ($\alpha = 1.0$, $k = 0.1$) as a starting point (matching numbers in your writeup).

(B) Initial condition (Gaussian-like wavepacket on torus)

- Standard test: a localized complex Gaussian

centered at $((\theta_0, \phi_0))$ with momentum (p_0) ,
normalized to unit (L^2) under $(\sqrt{g})$ measure.

- Use the same initial real-imag split for GSE:

$u = \text{Re}\psi_0, \quad v = \text{Im}\psi_0, \quad V = 0$ (or small random noise in (V) to see lift dynamics).

(C) Observables to compute at each time (t)

1. Global norm

$$N(t) = \int (u^2 + v^2) \sqrt{g} d\theta d\phi \quad \text{— check conservation.}$$

2. Expectation of position (on torus coordinates) — e.g. $(\langle\theta\rangle, \langle\phi\rangle)$ (use (sin, cos) to avoid wrapping).

3. Spatial variance (wavepacket spread) $(\sigma^2(t))$.

4. Energy: kinetic + potential (for standard Schrödinger);
for GSE you can define an analogous functional.

5. (V) -field metrics:

$$M_V(t) = \int V \sqrt{g}, \quad \max V, \quad \text{and spatial map } V(\theta, \phi, t)$$

6. Cross-correlation between two initially entangled locations (if testing synchronization) — but that's optional for the single-packet test.

(D) Numeric methods specifics

- Laplace–Beltrami: compute with spectral derivatives in $((\theta, \phi))$ (FFT) after factoring metric coefficients, or finite-difference on the grid using the divergence form:

$$\Delta_g w = \frac{1}{r(R + r \cos \theta)} \left[\partial_\theta \left(\frac{R + r \cos \theta}{r} \partial_\theta w \right) + \partial_\phi \left(\frac{r}{R + r \cos \theta} \partial_\phi w \right) \right]$$

- For GSE a split-step approach is natural:

a. Exact rotation step (pointwise):

$$(u, v) \mapsto (\cos(\Delta t/\hbar)u - \sin(\Delta t/\hbar)v, \sin(\Delta t/\hbar)u + \cos(\Delta t/\hbar)v)$$

OR incorporate rotation via RHS in RK4.

b. Hamiltonian step with CN (or explicit RK) using the Laplace–Beltrami.

c. (V)-update: integrate

$$\hbar \partial_t V = \alpha f \sqrt{u^2 + v^2} - kV$$

with an ODE solver (semi-implicit Euler is OK if (k) is not too small).

(E) Comparison metrics & expected differences

- Norm conservation:

- Standard Schrödinger should conserve

(N(t)) up to numerical error (CN gives good conservation).

- GSE: because of the added (V)-channel and possible dissipative terms, (N(t)) may still be conserved if no direct (u,v)-to-(V) sink is present in your modeling; check your chosen coupling carefully.

If amplitude shifts into (V) or is dissipated via (V), you will observe (N(t)) decrease.

- Wavepacket spreading:

- Standard Schrödinger: dispersive spreading

(Gaussian broadens unless bound by potential).

- GSE: rotation + lift tends to focus amplitude in toroidal layers (geometric focusing).
Expect reduced spreading and possible localization near curvature peaks (where f is large).

- Phase behavior:

- Standard: complex phase evolves, interference patterns form.

- GSE: the planar rotation still creates periodic behavior in (u, v) but because of (V) coupling and damping you will see

accumulation of (V) correlated with amplitude; interference fringes may be damped or reorganized.

- Energy:

- Standard Schrödinger: total energy conserved (again up to numerics).

- GSE: energy can flow into (V) and be dissipated by $(-k V)$; energy expectation will generally decay toward geometric equilibria unless you set $(k = 0)$.

- Qualitative hallmark: monitor $(V(\theta, \phi, t))$.

Under GSE you should see (V) grow where (ρ) overlaps large $(f(\theta, \phi))$ (curvature peaks), and then relax with rate (k) .

This is the unique signature of CUT-(i) dynamics.

(F) Example parameter set (start here)

$$\hbar = 1, m = 1, R = 10, r = 15.85, \alpha = 1.0, k = 0.5, N_\theta = 128, N_\phi = 256, \Delta t = 10^{-3}$$

(G) Diagnostics & plots to produce

- $(N(t))$ (norm) and energy vs time for both models.

- $(\sigma(t))$ (spatial spread) vs time.
- Snapshots of $(\rho(\theta, \phi, t))$ and $(V(\theta, \phi, t))$ at representative times.
- Time series at a point $((\theta_i, \phi_j))$ of $(u(t), v(t), V(t))$ to illustrate damped-helical motion.
- If you run an entanglement-like two-site initial condition:
correlation coefficient of (V) between sites.

(H) Expected outcomes summary

- Standard Schrödinger: unitary oscillatory propagation, dispersion, conserved norm and energy.
- GSE: rotation in (u, v) plus a driven, damped (V) channel that collects amplitude information and tends to geometrically focus amplitude;
norm/energy may transfer into/through (V) and dissipate (depending on model choices).
You should observe less dispersion and (V) -peaks correlated with curvature in GSE runs.